Research on the Solution of Equivalent Substitution for Undetermined Limit

Zhang Yihe, Wu Xiaolan, Zheng Jiannan
Fujian Electric Vocational and Technical College, Quanzhou, Fujian, 362000, China

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Abstract: This article provides the definition of comparison for infinite order, demonstrates several theorems of equivalent infinity, proves theorems of "$\frac{\infty}{\infty}$" and other equivalent substitution for undetermined limit, and uses these theorems' conclusions to solve corresponding undetermined limit. These methods are simple, convenient and easy for students to maser. Besides, these methods can stimulate interests of study, develop innovative thinking and foster science inquiry spirit of students to further lay a solid mathematical foundation for future study.

1. Introduction

Using theorem of equivalent substitution for infinitesimal to solve "$\frac{0}{0}$" limit is simple, convenient and easy for students to understand and maser. This article discusses "$\frac{\infty}{\infty}$" and other theorems of equivalent substitution for undetermined limit, and uses these theorems' conclusions to solve corresponding undetermined limit. These methods are simple, convenient, easy for students to maser, and largely increase speed of solving problems and accuracy. Moreover, these methods can develop divergent and innovative thinking of students, and foster their science inquiry spirit.

First, give the definition of comparison for infinite order.

Definition: suppose α, β is infinity of $x \to x_0$.

If $\lim\limits_{x \to x_0} \frac{\alpha}{\beta} = \infty$ then α is high-order infinity than β ;

If $\lim\limits_{x \to x_0} \frac{\alpha}{\beta} = 0$ then α is low-order infinity than β ;

If $\lim\limits_{x \to x_0} \frac{\alpha}{\beta} = c(c \neq 0)$ then $\alpha and \beta$ is same-order infinity; Especially, when c=1, $\alpha and \beta$ is equivalent infinity, recorded as $\alpha \overset{\infty}{\sim} \beta$.

Easily prove that $x \to +\infty, e^{\lambda x}(\lambda > 0)$ is higher order infinity than $x^n (n > 0)$; $x^n (n > 0)$ is higher order infinity than $\ln x$.

Obviously, variable α, β is equivalent infinitesimal of $x \to x_0$, then $\frac{1}{\alpha}, \frac{1}{\beta}$ is equivalent infinity of $x \to x_0$, and vice versa.

2. Theorems and inferences of equivalent infinity

Theorem 1: When $x \to x_0$, α is high-order infinity than β, then $\alpha \pm \beta \overset{\infty}{\sim} \beta$.

Prove: When $x \to x_0$, α is high-order infinity than β , so $\lim\limits_{x \to x_0} \frac{\beta}{\alpha} = 0$,

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then"" $\lim\limits_{x\to x_0}\frac{\alpha\pm\beta}{\alpha}=\lim\limits_{x\to x_0}\frac{1\pm\beta/\alpha}{1}=1$,therefore $\alpha\pm\beta\overset{\infty}{\sim}\beta$.

Theorem 2:When $x\to x_0$, α is infinity, c is constant, then $\alpha\pm c\overset{\infty}{\sim}\alpha$.

Prove: According to theme $\lim\limits_{x\to x_0}\frac{1}{\alpha}=0$, then $\lim\limits_{x\to x_0}\frac{\alpha\pm c}{\alpha}=\lim\limits_{x\to x_0}(1\pm c/\alpha)=1$,

therefore" $\alpha\pm c\overset{\infty}{\sim}\alpha$.

Inference 1: When $x\to x_0$, β_1、β_2、...β_k is always lower order infinity than α, c is constant, then $\alpha\pm\beta_1\pm\beta_2\pm...\pm\beta_k\pm c\overset{\infty}{\sim}\alpha$.

Inference 2: When $\alpha=a_n x^n+a_{n-1}x^{n-1}+...+a_1x+a_0,(a_n\neq 0)$, then $x\to\infty,\alpha\overset{\infty}{\sim}a_n x^n$.

Theorem 3:If when $x\to x_0$, α, βis equivalent positive infinity or equivalent positive infinitesimal, then $\mathrm{x}\to\mathrm{x}_0,\ln\alpha\overset{\infty}{\sim}\ln\beta$.

Prove: $\ln\alpha,\ln\beta$ is infinity

$$and\ \lim_{x\to x_0}\frac{\ln\alpha}{\ln\beta}=\lim_{x\to x_0}\frac{(\ln\alpha)'}{(\ln\beta)'}=\lim_{x\to x_0}\frac{\frac{1}{\alpha}(\alpha)'}{\frac{1}{\beta}(\beta)'}=\lim_{x\to x_0}\frac{\beta}{\alpha}\frac{(\alpha)'}{(\beta)'}$$

$$=\lim_{x\to x_0}\frac{\beta}{\alpha}\lim_{x\to x_0}\frac{(\alpha)'}{(\beta)'}=\lim_{x\to x_0}\frac{(\alpha)'}{(\beta)'}=\lim_{x\to x_0}\frac{\alpha}{\beta}=1.$$

then $\mathrm{x}\to\mathrm{x}_0,\ln\alpha\overset{\infty}{\sim}\ln\beta$.

3. Equivalent substitution theorem

Theorem 4:When $x\to x_0$, $\alpha,\beta,\alpha_1,\beta_1$ are infinity, and $\alpha\overset{\infty}{\sim}\alpha_1$, $\beta\overset{\infty}{\sim}\beta_1$, then $\lim\limits_{x\to x_0}\frac{\alpha}{\beta}=\lim\limits_{x\to x_0}\frac{\alpha_1}{\beta_1}$.

Prove: by $\alpha\overset{\infty}{\sim}\alpha_1$, $\beta\overset{\infty}{\sim}\beta_1$, , get:

$$\lim_{x\to x_0}\frac{\alpha_1}{\alpha}=\lim_{x\to x_0}\frac{\beta_1}{\beta}=1\Rightarrow\lim_{x\to x_0}\frac{\frac{1}{\alpha_1}}{\frac{1}{\alpha}}=\lim_{x\to x_0}\frac{\frac{1}{\beta_1}}{\frac{1}{\beta}}=1.$$

$gain:\frac{1}{\alpha}and\frac{1}{\alpha_1}are$ are equivalent infinitesimal, $\frac{1}{\beta}and\frac{1}{\beta_1}$ are equivalent infinitesimal.

By theorem of equivalent substitution for infinitesimal, obtain:

$$\lim_{x\to x_0}\frac{\frac{1}{\beta}}{\frac{1}{\alpha}}=\lim_{x\to x_0}\frac{\frac{1}{\beta_1}}{\frac{1}{\alpha_1}}\Rightarrow\lim_{x\to x_0}\frac{\alpha}{\beta}=\lim_{x\to x_0}\frac{\alpha_1}{\beta_1}。$$

4. " $0\cdot\infty$ "and" 1^∞ " equivalent substitution theorem

Theorem 5:When α and α_1 are equivalent infinitesimal, β and β_1 are equivalent infinity, then

$\lim\limits_{x\to x_0}\alpha\beta=\lim\limits_{x\to x_0}\alpha_1\beta_1.$

Prove: β and β_1 are equivalent infinity, soWhen $x\to x_0$, $\frac{1}{\beta}$ and $\frac{1}{\beta_1}$ are equivalent infinitesimal.

By theorem of equivalent substitution for infinitesimal, obtain:

$$\lim_{x\to x_0}\alpha\beta=\lim_{x\to x_0}\frac{\alpha}{\frac{1}{\beta}}\overset{\frac{0}{0}}{=}\lim_{x\to x_0}\frac{\alpha_1}{\frac{1}{\beta_1}}=\lim_{x\to x_0}\alpha_1\beta_1.$$

Theorem 6:When $x\to x_0$, α and α_1 are equivalent infinitesimal, β and β_1 are equivalent infinity, then $\lim\limits_{x\to x_0}(1+\alpha)^{\beta}=\lim\limits_{x\to x_0}(1+\alpha_1)^{\beta_1}$.

prove: $\lim\limits_{x\to x_0}(1+\alpha)^{\beta}=e^{\lim\limits_{x\to x_0}\alpha\beta}$, $\lim\limits_{x\to x_0}(1+\alpha_1)^{\beta_1}=e^{\lim\limits_{x\to x_0}\alpha_1\beta_1}$,

According to theme,by theorem 4 know that

$\lim\limits_{x\to x_0}\alpha\beta=\lim\limits_{x\to x_0}\alpha_1\beta_1$

$so\lim\limits_{x\to x_0}(1+\alpha)^{\beta}=\lim\limits_{x\to x_0}(1+\alpha_1)^{\beta_1}.$

Theorems 1 to 6 apply to same changing process of independent variable.

5. Examples of application

$Example1\qquad Solve\lim\limits_{x\to\infty}\frac{5x^4-3x^2+2x+1}{8x^5-4x^3-3x-2}$

Solution: by theorem 2 and inference 2 know

$$x\to\infty,5x^4-3x^2+2x+1\overset{\infty}{\sim}5x^4,8x^5-4x^3-3x-2\overset{\infty}{\sim}8x^5.$$

By theorem 4 get $\lim\limits_{x\to\infty}\frac{5x^4-3x^2+2x+1}{8x^5-4x^3-3x-2}=\lim\limits_{x\to\infty}\frac{5x^4}{8x^5}=\lim\limits_{x\to\infty}\frac{5}{8x}=0$

$Example2$ sole $\lim\limits_{x\to0^+}\frac{\ln[(\cos x+3)\tan 7x]}{\ln\tan 2x}$.

Solution: $x\to0^+$, by theorem 2 and inference 1 get:

$$\ln[(\cos x+3)\tan 7x]=\ln(\cos x+3)+\ln\sin 7x+\ln\cos 7x\overset{\infty}{\sim}\ln\sin 7x,$$

$$\ln\tan 2x=\ln\sin 2x+\ln\cos 2x\overset{\infty}{\sim}\ln\sin 2x.$$

by theorem4:

$$\lim_{x\to0^+}\frac{\ln[(\cos x+3)\tan 7x]}{\ln\tan 2x}=\lim_{x\to0^+}\frac{\ln\sin 7x}{\ln\sin 2x}$$

and $x\to0^+$, itesimal $\sin 7x\sim 7x,\sin 2x\sim 2x$。 by theorem3get:

$$\lim_{x\to0^+}\frac{\ln[(\cos x+3)\tan 7x]}{\ln\tan 2x}=\lim_{x\to0^+}\frac{\ln\sin 7x}{\ln\sin 2x}=\lim_{x\to0^+}\frac{\ln 7x}{\ln 2x}=\lim_{x\to0^+}\frac{\ln 7+\ln x}{\ln 2+\ln x}=\lim_{x\to0^+}\frac{\ln x}{\ln x}=1.$$

*Example*3 sole $\lim_{x\to 0^+}\dfrac{\ln\cot x}{\ln x}$.

solution: $x\to 0^+$, infinitesimal $\tan x\sim x, \Rightarrow \cot x \overset{\infty}{\sim} \dfrac{1}{x}$, by theorem 3 get:

$\ln\cot x \overset{\infty}{\sim} \ln\dfrac{1}{x}$.by theorem 4 get: $\lim_{x\to 0^+}\dfrac{\ln\cot x}{\ln x}=\lim_{x\to 0^+}\dfrac{\ln\frac{1}{x}}{\ln x}=\lim_{x\to 0^+}\dfrac{-\ln x}{\ln x}=-1.$

*Example*4 sole $\lim_{x\to 0}\sin 2x\cot 5x$.

solution: $x\to 0$, infinitesimal $\sin 2x\sim 2x$, $\tan 5x\sim 5x, \Rightarrow \cot 5x \overset{\infty}{\sim} \dfrac{1}{5x}$,

by theorem 5 get:

$$\lim_{x\to 0}\sin 2x\cot 5x=\lim_{x\to 0}\sin 2x\cot 5x=\lim_{x\to 0}2x\times\frac{1}{5x}=\frac{2}{5}.$$

Example5 sole $\lim_{x\to 0}(1+\arcsin 3x)^{\csc x}$.

Solution: $x\to 0$, infinitesimal $\arcsin 3x\sim 3x$, $\sin x\sim x, \Rightarrow \csc x \overset{\infty}{\sim} \dfrac{1}{x}$,

by theorem 6 get: $\lim_{x\to 0}(1+\arcsin 3x)^{\csc x}=e^{\lim_{x\to 0}3x\times\frac{1}{x}}=e^3.$

Through these examples of application, we can find out that comparing with common solution, using theorem of equivalent substitution is simple, convenient and easy to understand and master, and largely increases peed of solving problems and accuracy. Meanwhile, using this solving method can develop trains of thought and innovative spirit of students, foster study spirit and increase learning interests and comprehensive qualities to further lay a solid mathematical foundation for future study.

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References

[1] Department of Applied Mathematics, Tongji University, Advanced Mathematics (First Volume), Beijing: Higher Education Press, 2004.

[2] Zhang Yi-he. Research on the Solution for the Limits of Type "1^∞". Science and Wealth, 2016(11): 865